Seven is the Magic Number in Nature

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> Joshua and Israel marched around Jericho seven times while seven priests blew seven trumpets before the walls came crashing down. (Joshua 6:3-4)

Introduction

Where there is structure, the parts of the structure must function together with a degree of consistency and purpose. Specifically, I am thinking of dynamic systems in which there are action and reaction among the parts and their functions and also friction and resistance. Natural systems, such as the cells in our bodies, and man-made systems, such as a watch, are constructed in a hierarchic way so that the different parts in each level work together consistently—that is, each group performs a function to fulfill some purpose. Thus, the number of functions working together determines the structure through which materials or energy pass. The number of functions that can work together is determined by the consistency of the interactions of these functions. Conversely, consistency among the functions depends on the number of interacting components; if there is a large number, the possibility of inconsistency is greater. How large should the number of functions be to fulfill a purpose? The answer given here has important implications for constructing both physical and social systems. The current paper shows with mathematics supported by examples that 7 to 8 seem to be the maximum number for any component of a complex system.

A system consists of a structure, flows in the structure, functions or actions that the flows perform, and a purpose for the system to fulfill. There can be multiple flows, functions, and purposes served. For example, to survive the human body must perform a few interacting functions through its flows, such as circulating blood, breathing, digesting, reproducing, sending hormones, firing nerves, moving muscles, obtaining support from bones, and relying on integumentary parts (e.g., hairs, nails). The last two or three serve to support the functions of the other organs and are fairly independent of them.

The functions themselves are a synthesis of lesser functions; digestion involves chewing, tasting, swallowing, secreting chemicals, breaking down complex sugars into simple sugars and proteins into amino acids, and emulsifying fats, absorbing the nutrients of the food we eat, and excreting the waste. These sub-function themselves can each be broken down to lesser sub-functions. Thus, the structure of any system needs to be broken down hierarchically into modules to facilitate the flows in that system and their functions. Modularity is a general principle for managing complexity. By breaking down a complex system into discrete pieces—which can then communicate with one another only through standardized interfaces within a standardized architecture—one can eliminate what would otherwise be an unmanageable tangle of system-wide interconnections.

The functions interact and depend on each other—each one of them is important for the maintenance and survival of the other functions. However, for a system or subsystem to survive, there cannot be an excessive number of functions. Such an idea is not new in the literature of technological design (Simon, 1962).[1] The aforementioned theory is thought to have been operating as a law of nature from the beginning even if, as some claim (Baldwin and Clark, 1997), modularity is becoming more important today because of the increased complexity of modern technology. We can apply the idea of modularity not only to technological design but also to social organizations.

The structure of a system is designed to accommodate certain flows that pass through it. Subsystems of the system have different functions that interact, which lead to the fulfillment of the overall purpose. The functions must therefore work together (i.e., be interdependent and conjoint and give feedback) to achieve the purpose. When one or more functions are faulty, the purpose the system is designed to serve fails in different degrees. The functions can take place sequentially, conjointly, or in combination. When they are sequential, as in a relay race, there is no problem in being consistent (except perhaps in handing the baton). The important question to be examined in the current paper is how consistently interdependent functions combine to achieve the desired purpose.

For the Nobel Laureate Herbert Simon,[1] a complex system is:

> . . . one made up of a large number of parts that interact in a non-simple way. In such systems, the whole is more than the sum of the parts, at least in the important pragmatic sense that, given the properties of the parts and the laws of their interaction, it is not a trivial matter to infer the properties of the whole.

Simon also says "complexity is both a matter of the sheer number of distinct parts the system comprises and of the nature of the interconnectedness among those parts."

Looking at it differently, however, modularity has an even longer pedigree in the social sciences. We can think of the "obvious and simple system of natural liberty" in Adam Smith's *Wealth of Nations* (1776), where he showed that a complex modern society with its social and economic institutions needs modular design to be more productive.[2, 3]

A hierarchy is one of two ways to structure a system that is composed of interrelated subsystems that are each hierarchic in turn. The other way to structure a system is as a network. In formal organizations, the number of subordinates who report directly to a single boss is called his or her "span of control." Analogously, the span of a system is the number of subsystems into which it is partitioned. Simon[1] says that a hierarchic system is flat at a given level if it has a wide span at that level. A diamond has a wide span at the crystal level but not at the next level down (i.e., the molecular level).

One important difference exists between physical and biological hierarchies, on the one hand, and social hierarchies, on the other. Most physical and biological hierarchies are described in spatial terms. We detect the organelles in a cell in the way we detect the raisins in a cake—they are "visibly" differentiated substructures localized spatially within the larger structure. In social hierarchies, one considers who interacts with whom, not who lives next to whom. The width of span in a hierarchic system is of concern in this paper.

We are not thinking of "dead" parts, such as the wires in circuits that conduct electricity to destinations. There can be millions of them. Also we are not thinking about collections of objects arranged in orderly ways to form a structure. We are thinking about objects that are dynamic and function together according to natural or manmade forces that act to fulfill a purpose. None of the parts can function well or at all without the presence of the others, as, for example, in the case of a car's cylinders or a clock's wheels, and within natural organisms (i.e., the parts or organelles of a living cell that need one another to survive). In the case of organelles, interactions are not mechanically direct but rather act through chemistry and a medium, the cytoplasm. The organs of our body use the circulatory system and the blood supported by the materials they produce to help nurture each other and the rest of the body. They all work together and influence each other—they are interdependent in performing their function. The effect on the organism may take a longer time to manifest these influences, good or bad; their influence may take a shorter time to be felt and noticed. If we stop breathing, it can be the end of us because of the lack of oxygen as the heart stops pumping blood to the brain and other organs.

Underlying this interdependence and feedback is the degree of consistency or harmony in the interaction of the functions. Consistency in the workings of the parts of the system determines the degree of stability of the system. Inconsistency can lead to instability and to the system faltering and ceasing to function. Inconsistency varies in intensity from extremely inconsistent, to randomly inconsistent, to moderately inconsistent, and, finally, to perfectly consistent. There can be measurements associated with the degree of inconsistency with which any system of multiple parts and functions is operating. It is possible that there could also be underlying simple laws of form, which a rational mind might apprehend to explain complexity.

The philosopher, Arthur Schopenhauer said, "Every truth is the reference of a judgment to something outside it, and intrinsic truth is a contradiction." Since there are no absolutes, comparisons must be used, which inevitably lead to judgments and the possibility of inconsistency because of the subjectivity and variability of judgments. When we deal with intangible factors, which by definition have no scales of measurement, we can compare them in pairs according to the dominance of one over another with respect to a common property. We can not only determine the preferred object but also discriminate among intensities of preference. When we compare functions, each working in its own domain, we can compare how well a function is doing with how well it was doing before. The common property in such a comparison is how well it fulfills its purpose. But when we compare two functions, what is the common property? They may have very different purposes. The common property needs to be, as also affirmed by Simon, some emergent entity that comes from their interaction, "contributing to maintaining synchronous timing?" for example. Another possibility to compare functions pairwise within a given hierarchical level could be: "With respect to the higher purpose (a node/function in the above higher level) which function better defines, and to what extent, this higher purpose?"

Why Consistency is Essential for the Workings of any System

The Oxford English dictionary defines inconsistency as a "want of agreement or harmony between two things or different parts of the same thing." Webster's dictionary defines consistency as "agreement or harmony in parts or of different things." This definition is the commonsense view of consistency, but there is also a mathematical version of consistency derived by considering the elements of the system in pairs.

Consistency forms the basis of causal thinking, but it also applies to the workings of things, as the dictionary says. The insightful Julian Huxley[4] wrote that "something like the human mind might exist even in lifeless matter." Herms Romijn[5] has published a substantial paper in which he argues persuasively that photons have consciousness. The article suggests that photons carry subjectivity or consciousness as a given property, which is possible in principle because irreducible properties (nothing is smaller than a photon) are present at this level. He argues that it is more reasonable than the current approach, which suggests that the new property of consciousness can be produced by banging together previously unconscious bits of matter.

It is with the consistent interaction of functions that the purpose is fulfilled. If the functions are inconsistent, the purpose is less perfectly satisfied. The question is: What should the number of functions be, and what level of inconsistency can the purpose tolerate before it begins to show signs of deterioration?

To be consistent is not to lead to contradictions. This definition is independent of time. When a system is dynamic and depends on time, the foregoing definition of consistency involves time in a different manner. Is there consistency or harmony among the parts of the system so they continue to work together? How bad can inconsistency be? If we are close to consistency, we expect that the system will continue to function well. That closeness to consistency is sufficient because no system is perfectly consistent.

To say that A is twice as heavy as B and B is 3 times as heavy as C and conclude that A is 6 times heavier than C is a consistent way of thinking. If one were to conclude that A is 5 times as heavy as C, one would think it is not as wrong as saying A is 100 times as heavy as C. Consistency in language means that reasoning does not lead to contradictory outcomes, and this example is a mathematical way to express that idea.

More about Consistency and Inconsistency in Science, Mathematics, and Engineering

The idea of consistency, with some exceptions, is not used much in philosophy or mathematics. One speaks of the consistency of a set of axioms in that they do not produce contradictory results. When a set of equations are all satisfied by at least one set of values for the variables, they are said to be consistent. If they are not all satisfied by any one set of values for the variables, they are said to be inconsistent. We also assume that the real world is consistent, and it is our job to describe it in a consistent way. But even in physics, it does not always happen; the theory of relativity and quantum theory have not been reconciled in a consistent way.

When we say that the cylinder of an engine is inconsistent in its function with its intended design, we mean that one cylinder is not functioning as closely to its design as other cylinders may be. If we compare the relative inconsistency of these cylinders with the intended design, we would say that this cylinder behaves "equally as well," "a little better," or "strongly better" than the other. In the end, we can obtain a measure of the priorities of the cylinders according to their consistency with their design.

This example can be generalized to any system, and we can use the same measure of consistency. We note that it is easier to compare the cylinders among themselves for the degree of consistency because we can observe them. This approach is clearer than comparing each with the design ideal that it is assumed to fulfill, since one may have little information about the design and its implementation. By comparing the cylinders with one another, despite their inconsistent functioning, we can seek better performance of the system of cylinders. It is clear that the more consistent cylinders tend to compensate for the inconsistency of the less consistent ones. This kind of interdependence is what we are referring to. In addition, many inconsistent cylinders can cause the good cylinders to become less consistent in their attempt to improve performance. It should now be clear that if the number of cylinders is large and many of them are inconsistent, the compensation of the other cylinders becomes less sensitive, and the system now gradually slows down in attaining its purpose, which is the reasoning behind the measurement of inconsistency. Of course, the number of cylinders can also be small but inconsistent. However in this case, one can identify the most inconsistent cylinder and attempt to repair it. Many inconsistent cylinders cause "overcompensation" by the consistent cylinders, which can wear out/decrease performance.

The same ideas apply in social situations. The members of a jury can be considered to reason with the facts. However, some of them are better at drawing conclusions of "guilty" or "not guilty" from these facts. When we compare them, we find that some jurors seem to be more inconsistent than others. A large number of jurors prevents us from determining which of the jurors are inconsistent in their treatment of the facts. In biology, the internal organs of the body depend on each other's functioning for their survival; the more organs there are, the more difficult it becomes for them to compensate for inconsistency in other organs. If they compensate strongly, the system will become dysfunctional because they respond only with their own chemistry and material to balance the system.

Let us now examine in greater depth the idea of consistency and inconsistency in our minds. Because inconsistency also occurs in nature according to the definition given earlier, the ideas developed below can also be generalized to inconsistency in nature. A good example of inconsistency, in the world of sports, is that team A beats team B, team B beats team C, but team C beats team A. Here, inconsistency is a natural occurrence and not a mental aberration. It is surprising that we create axioms for economics that preclude such intransitivity, which abounds in people's expression of many different kinds of preference. It is easy to identify inconsistency in simple situations such as the one noted above: if A is 3 times more important than B and B is 2 times more important than C, then A should be 6 times more important than C, not 5 times. Here, the logical approach works well, but if there are many things to compare, it becomes difficult to be perfectly consistent in making judgments. Thus how many elements there are influences tracking the consistency of the system, whether in our mind's thinking or in physical systems of the natural world.

To emphasize the point, interacting with consistency means to work together in harmony, agreement, or concord to fulfill a purpose. There is a cogent, logical, and mathematical reason to decompose any complex system with interactive parts. The clearest way to deconstruct the system is in a hierarchical fashion, breaking it down into small groups or levels of homogeneous parts.[6] These parts must be "similar," or more precisely be of the same "order of magnitude," to work together consistently.

Hierarchic Decomposition of a System: The Role of Modularity to Allow Different Flows to Serve Different Functions

Simon[1] introduces the topic of evolution of a complex system with a parable. There once were two watchmakers named Hora and Tempus who manufactured very fine watches. Both of them were highly regarded, and the phones in their workshops rang frequently; new customers were constantly calling. However, Hora prospered whereas Tempus became poorer and poorer and finally lost his shop. What was the reason? The watches the men made consisted of about 1,000 parts each. Tempus had constructed his so that if he had one partly assembled and had to put it down (to answer the phone, for example) it immediately fell to pieces and had to be rebuilt. The more the customers liked his watches, the more they phoned him. Therefore, it became difficult for him to find enough uninterrupted time to finish a watch.

The watches that Hora made were no less complex than those of Tempus. But he had designed them so that he could put together subassemblies of about 10 elements each. Ten of these subassemblies, again, could be put together into a larger subassembly; and a system of 10 of the latter subassemblies constituted the whole watch. Hence, when Hora had to put down a partly assembled watch to answer the phone, he lost only a small part of his work and he assembled his watches in only a fraction of the man-hours it took Tempus.

It is rather easy to make a quantitative analysis of the relative difficulty of the tasks of Tempus and Hora. In the end, however, what makes Tempus's unfinished watches so unstable is not the sheer number of distinct parts involved; Rather, it is the *interdependency* among the parts in his design that cause the watches to fall apart. In a non-decomposable system, the successful operation of any given part is likely to depend on the characteristics of many other parts throughout the system. So when such a system is missing parts (because it is not finished, for example, or because some of the parts are damaged), the whole ceases to function.

By contrast, in a decomposable system, the proper working of a given part will depend highly on the characteristics and consistency of other parts within its subassembly—but it will also depend on the characteristics and consistency of parts outside of that subassembly. As a result, a decomposable system may be able to limp along even if some subsystems are damaged or incomplete. In organizational and social systems—and even in mechanical ones as well—it is possible to think of interdependence and interaction among the parts as a matter of information transmission or communication. The flows in the system are the means of communication, whether mental or physical, like opening a car door with a remote control. Communication of information requires consistency with existing knowledge or, more abstractly, with the function of an existing structure. Consistency is the most important criterion in building information. The test for consistency is comparative. In fact, no absolute method for testing the consistency of a set of assumptions has ever been found. For a mathematical definition of consistency in a hierarchy see Saaty (2010).[7]

A good example of a decomposable system is the central nervous system made of nerve cells and their interactions. It is made up of the brain, the spinal cord, and the peripheral nervous system. The brain is made of three main parts: the forebrain, midbrain, and hindbrain (Figure 1). The forebrain consists of the cerebrum (little brain), associated with higher brain function such as thought and action; the thalamus; and the

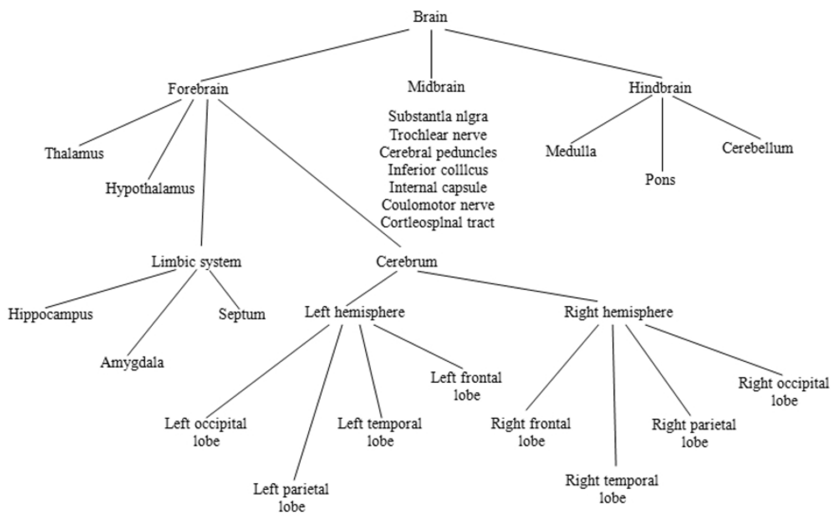

FIGURE 1. A simplified hierarchic model of the brain.

hypothalamus (part of the limbic system or "emotional brain"). The midbrain consists of the tectum and the tegmentum. The hindbrain is made of the cerebellum, pons, and medulla. Often, the midbrain, pons, and medulla are referred to together as the brainstem, which is responsible for basic vital life functions such as breathing, heartbeat, and blood pressure. The spinal cord is made of a bundle of nerves running up and down the spine. The peripheral nervous system consists of the nerves and ganglia outside of the brain and spinal cord. Its function is to connect the central nervous system to the limbs and organs. It is divided into the somatic nervous system and the autonomic nervous system, including sensory systems. Most of the 12 cranial nerves are part of it. Figure 1 shows that we have a complex hierarchic structure that again needs overall measurement of the consistency of its functions, which is developed later in the current paper.

How to Measure Inconsistency

What is known in the theory of measurement[6] as the Fundamental Scale is used to make paired comparison judgments. This scale can be derived from the logarithmic stimulus-response function of Weber-Fechner in psychophysics.[7]

In a less mathematical vein, we are able to distinguish between high, medium, and low at one level, and for each of them in a second level below, we also are able to distinguish between high, medium, and low for each of the three giving us nine different categories. We assign the value 1 to (low, low), which is the smallest, and the value 9 to (high, high). Doing so allows us to cover the spectrum of possibilities between two elements and to give the value nine for the top of the paired comparisons scale compared with the lowest value on the scale (validation examples of this scale appear later). The mathematician and cognitive neuropsychologist Stanislas Dehaene[8] writes: "Introspection suggests that we can mentally represent the meaning of numbers 1 through 9 with actual acuity. Indeed, these symbols seem equivalent to us. They all seem equally easy to work with, and we feel that we can add or compare any two digits in a small and fixed amount of time like a computer." The fundamental scale in measurement theory comprises the numbers 1 to 9.

In making paired comparisons, numbers are assigned to pairs of elements using judgment about dominance. The elements being compared must be homogeneous, requiring no greater number than 9. An element compared with itself with respect to a certain criterion is always equal to 1. Therefore, the main diagonal entries of the pairwise comparison matrix are all 1. The numbers 3, 5, 7, and 9 correspond to the verbal judgments "moderately more dominant," "strongly more dominant," "very strongly more dominant," and "extremely more dominant," with 2, 4, 6, and 8 between the previous values. Reciprocal values are automatically entered in the transpose position. We are permitted to interpolate values between the integers, if desired, or use numbers from an actual ratio scale of measurement. The decision-making theory known as the Analytic Hierarchy Process (AHP) uses the integers 1 to 9 as its Fundamental Scale of Absolute Numbers corresponding to the aforementioned verbal statements for the comparisons.

The ordinary way of measuring tangibles uses scales that have a unit to measure and assigns a number to each object one at a time. In the paired comparisons process, a number is assigned not to the objects but to the relation of dominance between two objects or functions at a time. A priority scale is then derived from all the dominance measurements for the objects. In this manner, we are able to both derive priorities for tangibles, and for intangibles for which there is no measurement. Although numbers obtained by using a scale are permanent and are always the same, priorities are only useful for the problem at hand (i.e., the group of objects being compared) and need not be the same for another problem with the members of the group changed.

Let $A = (a_{ij})$ be an n-by-n positive reciprocal matrix, so all $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$, for all i, j $1,\ldots,n$. Let $w = (w_1,\ldots,w_n)$ be the principal right eigenvector of $A, \sum_{i=1}^{n} w_i = 1$ which captures the priority of "dominating" for each element in the group, and let $v = (v_1, \ldots, v_n)$ be the principal left eigenvector of A, $\sum_{i=1}^{n} v_i = 1$, which captures the priority of "being dominated." Now if a dominance matrix is consistent, then one can write its entries as ratios of its priority vector entries so that $a_{ij} = w_i / w_j$.

Because in practice A is usually inconsistent, we write $a_{ij} = \varepsilon_{ij} w_i / w_j, \varepsilon_{ij} > 0$. Moreover, $\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i$, $i = 1, \ldots, n$, and substituting for a_{ij}, we have $\sum_{j=1}^{n} \varepsilon_{ij} = \lambda_{\max}$.

The computation:

$$n\lambda_{\max} = \sum_{i=1}^{n} (\sum_{j=1}^{n} \varepsilon_{ij}) = \sum_{i=1}^{n} \varepsilon_{ii} + \sum_{1 \le i < j \le n} (\varepsilon_{ij} + \varepsilon_{ji}) = n + \sum_{1 \le i < j \le n} (\varepsilon_{ij} + \varepsilon_{ij}^{-1}) \ge n + 2(n^2 - n)/2 = n^2$$

reveals that $\lambda_{\max} \ge n$. Moreover, since $x + 1/x \ge 2$ for all $x > 0$, with equality if and only if $x = 1$, we see that $\lambda_{\max} = n$ if and only if all $\varepsilon_{ij} = 1$, which is equivalent to having all $a_{ij} = w_i / w_j$ obtained when the judgments are consistent.

The priorities are obtained by raising the matrix to arbitrarily large powers to capture the transitivity of dominance along chains of arbitrarily large lengths.[6] Let us now introduce a measure or index for inconsistency, or the deviation of $\lambda_{\max}$ from n. We represent the consistency index *by*: $\mu \equiv \frac{\lambda_{\max} - n}{n - 1}$.

Thus $\mu \ge 0$. $\mu = 0$ if and only if A is consistent. What about the term "n-1" in the denominator? Because the elements on the main diagonal are each equal to 1, their sum is equal to n. If we denote the eigenvalues of A that are different from $\lambda_{\max}$ by $\lambda_2, \ldots, \lambda_n$, it is known that the sum of

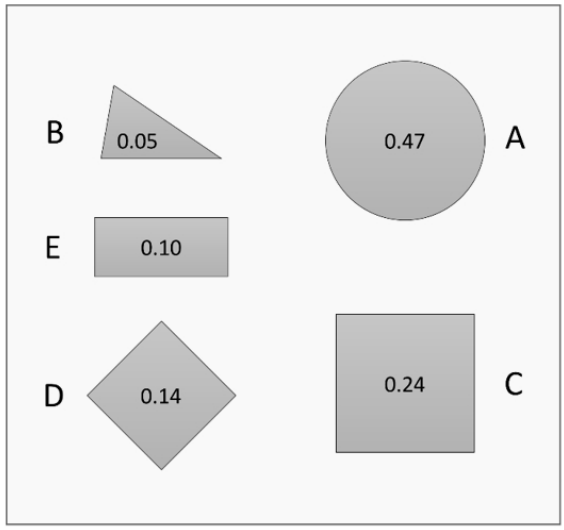

FIGURE 2. Area example.

all the eigenvalues of A is equal to the sum of the elements down the main diagonal, which is equal to n in this case. We have $n = \lambda_{\max} + \sum_{i=2}^{n} \lambda_i$, so $n - \lambda_{\max} = \sum_{i=2}^{n} \lambda_i$ and $\mu = -\frac{1}{n-1}\sum_{i=2}^{n} \lambda_i$ is the negative average of the non-principal eigenvalues of A.

We said before that we need a large order matrix to improve the validity of our results by reproducing answers that correspond to the real underlying answer. The following section will attempt to determine how large the matrix should be.

VALIDATION

To illustrate that this approach is not a number-crunching scheme but rather relates closely to the reality of actual measurement, consider a person who would like to estimate the relative area of the five geometric shapes given in Figure 2. It is an example of measurement with respect to a tangible criterion. For the purpose of this illustration, the relative area inside each shape obtained from actual measurement is also given. Of course, in real life situations, the relative areas would not be known to the person. He or she needs to estimate the relative sizes of the figures by comparing them in pairs. A pairwise comparison consists of

Figures	Circle	Triangle	Square	Diamond	Rectangle	Priorities from comparisons	Actual relative size
Circle	1	9	2	3	5	0.457	0.471
Triangle	1/9	1	1/5	1/3	1/2	0.049	0.050
Square	1/2	5	1	2	3	0.257	0.234
Diamond	1/3	3	1/2	1	2	0.150	0.149
Rectangle	1/5	2	1/3	1/2	1	0.087	0.096

Inconsistency = 0.003

TABLE 1. Matrix of judgments, outcomes, and actual relative sizes of the five geometric shapes.

	Drinks	Coffee	Wine	Tea	Beer	Sodas	Milk	Water	Derived priorities	Actual relative consumption
	Coffee	1	9	3	1	1/2	1	1/2	0.142	0.133
	Wine	1/9	1	1/3	1/9	1/9	1/9	1/9	0.019	0.014
A=	Tea	1/3	3	1	1/4	1/5	1/4	1/5	0.046	0.040
	Beer	1	9	4	1	1/2	1	1	0.164	0.173
	Sodas	2	9	5	2	1	2	1	0.252	0.267
	Milk	1	9	4	1	1/2	1	1/2	0.148	0.129
	Water	2	9	5	1	1	2	1	0.228	0.240

TABLE 2. Relative consumption of drinks.

identifying the figure with the smaller area of the two and estimating numerically how many times larger the area of the larger figure is than the area of the smaller one using the fundamental scale. The smaller figure is then assigned the reciprocal value when compared with the larger one. These comparisons are arranged in a five-by-five matrix as illustrated in Table 1. Conventionally, the item on the left side of the matrix is compared with that on top. If it is larger, the whole number corresponding to the judgment is put in that cell. If it is smaller, the reciprocal value is put in the cell.

A second example is about estimating relative drink consumption in the United States. To make the comparisons, the types of drinks are listed on the left and at the top of Table 2, and judgment is made as to how strongly the consumption of a drink on the left dominates that

Order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Random Index (RI)	0	0	0.5236	0.8835	1.1094	1.2496	1.3406	1.4039	1.4499	1.4854	1.5136	1.5355	1.5547	1.5709	1.5834
First Order Differences		0	0.5236	0.3599	0.2258	0.1402	0.0910	0.0633	0.04600	0.0355	0.0282	0.0219	0.0192	0.0162	0.0125

Table 3. Random index.

of a drink at the top. At the right of Table 2, it is apparent that the derived values and the actual values (obtained from various pages of Statistical Abstract of the United States) are close by nearly any measure of closeness.

The theory itself also provides us with a compatibility index between the derived and actual results without the need for statistical theory. We denote by $x=(x_i)$ and $y=(y_j)$, respectively, the derived and actual scale priorities, and by $c=(c_{ij})$ the matrix of $c_{ij}=(x_i/x_j)(y_j/y_i)$ of one matrix of ratios of the two scales and the transpose of the other matrix of ratios. We then sum all the elements of C and divide by n^2 to obtain the Compatibility Index, a number that represents the deviation from perfect consistency of the two vectors. The index for the drinks example is 1.036. If the two vectors were identical, the index would be 1. The less compatible they are, the higher the value will be above 1.

Statistical Demonstration of the Leveling Off of Inconsistency as the Number of Elements Increases

As the number of elements becomes too large, we will show that the inconsistency levels off, and it becomes literally impossible to use it to diagnose the faulty elements. Computers can do the tedious work of double-checking logical proofs, even for very large order matrices. To get some feel for what the consistency index might be telling us about a positive *n*-by-*n* reciprocal matrix A, consider the following simulation: choose the entries of A above the main diagonal at random from the 17 values of the Fundamental Scale {1/9, 1/8,…,1, 2,…,8, 9}. Then fill in the entries of A below the diagonal by taking reciprocals. Put 1's down the main diagonal and compute the consistency index. Do this 50,000 times and take the average, which is the random index. Table 3 shows the values obtained from one set of such simulations and also their first and second order differences, for matrices of size 1, 2,…,15.

Since it would be pointless to try to discern any priority ranking from a set of random comparison judgments, we should be uncomfortable proceeding unless the consistency index of a pairwise comparison

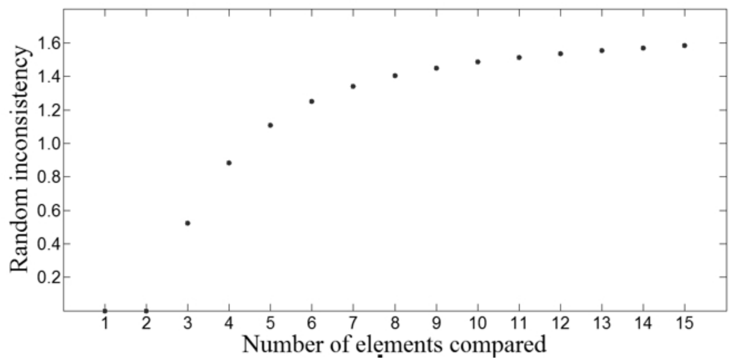

FIGURE 3. Plot of random index.

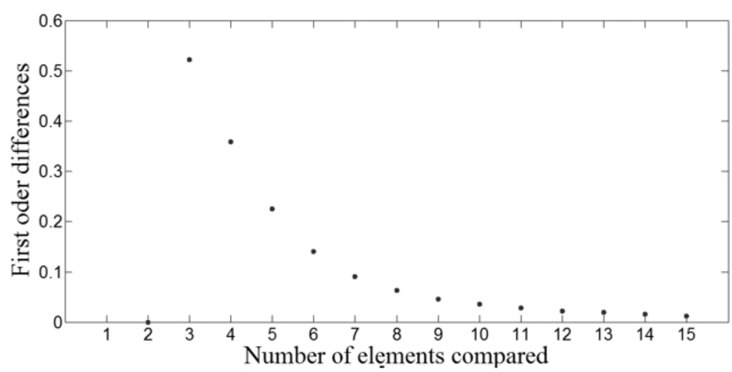

FIGURE 4. Plot of first order differences.

matrix μ is much smaller than the corresponding random index value in Table 3. The consistency ratio (CR) of a pairwise comparison matrix is the ratio of its consistency index to the corresponding random index value (RI) in Table 3.

Figure 4 is the plot of RI and shows the importance of the number 7 when taking differences of the random values of Figure 3 as the graph levels off when the number of elements is around 7. The number 8 is a cutoff point beyond which the differences are less than 0.10. It follows that the CR is no longer meaningful when many more elements are being compared in the matrix because the differences are small (0.05 or less), meaning that the RI values in the second row of Table 3 get too close past n = 7 to provide useful information for the CR to be used determine the most inconsistent judgment.[a]

a In addition, although the RI levels off as n increases beyond 8 (and the first order differences become negligible beyond n = 7), the matrix consistency index will most likely keep increasing beyond n = 7 (the more elements, the more difficult to be consistent), leading

The notion of "order of magnitude" is essential in any mathematical consideration of changes in measurement. For example, when one has a numerical value between 1 and 10 for some measurement and wants to determine whether change in this value is significant or not, the following line of reasoning can be used. A change of a whole integer value is critical because it changes the magnitude and identity of the original number significantly. If the change in value is a percent or less, it would be small (by two orders of magnitude) and thus would be considered negligible. However if this perturbation is a decimal (one order of magnitude smaller), we are likely to modify the original value by this decimal without losing the significance and identity of the original number as we first understood it to be. Thus in synthesizing near consistent judgment values, changes that are too large can cause dramatic change in our understanding, and values that are too small cause no change in our understanding. We are left with only values of one order of magnitude smaller that we can deal with incrementally to change our understanding. It follows that our allowable CR should be no more than about 0.10. The requirement of 10% cannot be made smaller, such as 1% or 0.1%, without trivializing the impact of inconsistency. Assuming that all knowledge should be consistent contradicts the experience that requires continued updating of understanding.

If the CR is larger than desired, we do three things: (1) find the most inconsistent judgment in the matrix (for example, the judgment for which $\varepsilon_{ij} = a_{ij}\, w_j / w_i$ is largest); (2) determine the range of values to which the judgment can be changed so that the inconsistency would be improved; and (3) ask the judge to consider changing his or her judgment to a plausible value in that range. If he or she is unwilling, we try with the second most inconsistent judgment and so on.

If no judgment is changed, the decision is postponed until better understanding of the stimulus is obtained. In my experience in decision making in numerous applications, judges who understand the theory are always willing to revise their judgments partially and subsequently examine the second most inconsistent judgment and so on. It can happen that a judge's knowledge does not permit one to improve his or her consistency, and more information is required to improve the consistency of judgments.

To see how large random inconsistency can get in using the Fundamental Scale, experiments were made with 3 x 3 up to 9 x 9 matrices with 9 and 1/9 alternating in each row and column, for matrices of

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to a more inconsistent system. In conclusion, beyond n = 7, it is not possible (at least from a practical point of view) to adjust any inconsistency, while at the same time it is also much easier to be inconsistent.

order 3, 5, 7, and 9 yielded the following respective results for λ_{max}: 10.1111, 19.2222, 28.3333, and 37.4444 rounded off to four decimal places. For successive odd powers of a matrix, one can see that λ_{max} increased by the same amount 9.1111 from one matrix to the next.

The quality of response to stimuli is determined by three factors: accuracy or validity, consistency, and efficiency. Our judgment is much more sensitive and responsive to large perturbations. When we speak of perturbation, we are referring to numerically changing consistent ratios. Conversely, the smaller the inconsistency, the more difficult it is for us to know where best the changes should be made. Once near consistency is attained, it becomes uncertain which coefficients should be perturbed by small amounts to transform a near consistent matrix to a consistent one.

Inconsistency and Perturbation of the Principal Eigenvector of Priorities

By using the inconsistency index for a pairwise comparison matrix, we have seen that the maximum numbers are 7 or 8. Now we use perturbations of the principal eigenvector w_1 to show that a similar result is obtained.

We have two formulas for representing the perturbations; the first is thanks to Wilkinson[9] for a general matrix, and the second for positive reciprocal matrices, which is easier to use.[10]

$$\Delta w_1 = \sum_{j=2}^{n} (v_j^T \Delta A w_1 / (\lambda_1 - \lambda_j) v_j^T w_j) w_j \quad (1)$$

$$\Delta w_1 = w'_1 - w_1 \quad (2)$$

In the first formula, w_1 represents the principal eigenvector (priority vector), and the other w's represent the remaining right eigenvectors of matrix A, whereas the v's represent the left eigenvectors of A and the λ's represent its eigenvalues. We use the first formula only to show that n must be small. We then use the second formula in a large number of simulations to show that the maximum value of n is 7 or 8 depending on how sensitive and responsive the system is to inconsistency.

In the first formula, we note that the eigenvector (priority vector) w_1 will be very sensitive to perturbations in A if λ_1 is close to any of the other eigenvalues. When λ_1 is well separated from the other eigenvalues and none of the products $v_i^T w_i$ of left eigenvectors v_i and right eigenvectors W_i is small, the eigenvector w_1 corresponding to the eigenvalue λ_1 will be comparatively insensitive to perturbations in A.

Number of Elements	Final Average Value of the Norms of $\Delta w_1 = w'_1 - w_1$ in the 50,000 Size Sample	First Order Difference of the Values in the 2nd Column
2	0.3085	
3	0.3048	0.0037
4	0.2961	0.0087
5	0.2742	0.0219
6	0.2498	0.0244
7	0.2242	0.0256
8	0.2016	0.0226
9	0.1816	0.0200
10	0.1651	0.0165
11	0.1507	0.0144
12	0.1385	0.0122
13	0.1277	0.0108
14	0.1188	0.0089
15	0.1109	0.0079

TABLE 4. Perturbation and first order differences.

The $v_i^T w_i$ are interdependent in a way that precludes the possibility that just one $1 / v_i^T w_i$, $i = 1,\ldots,n$ is large. Thus if one of them is arbitrarily large, they are all arbitrarily large. However, we want them to be small, i.e., near 1. The eigenvector w_1 is stable when:

(1) The perturbation ΔA is small as the consistency index might suggest;
(2) λ_j is well separated from λ_1; when A is consistent, $\lambda_1 = n$, $\lambda_j = 0$;
(3) The product of left and right eigenvectors is not excessively large, which is the case for a consistent (and near-consistent) matrix if the elements are homogeneous (compared here on the relative dominance scale of the Fundamental Scale) with respect to the criterion of comparison; and
(4) If the number of their entries is small.[1]

We note that n, the order of the matrix, should not be overly small because then one does not get enough information from the few comparison judgments to obtain valid results for real world measurement. To determine the magnitude of n, we need to examine the effect of random inconsistency on the order n of a positive reciprocal matrix A that would influence the number of terms in the sum defining Δw_1.

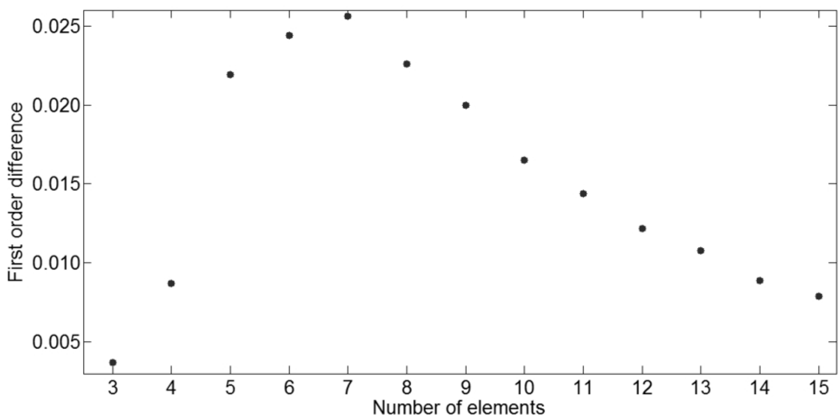

FIGURE 5. Plot of perturbations in second column of Table 4.

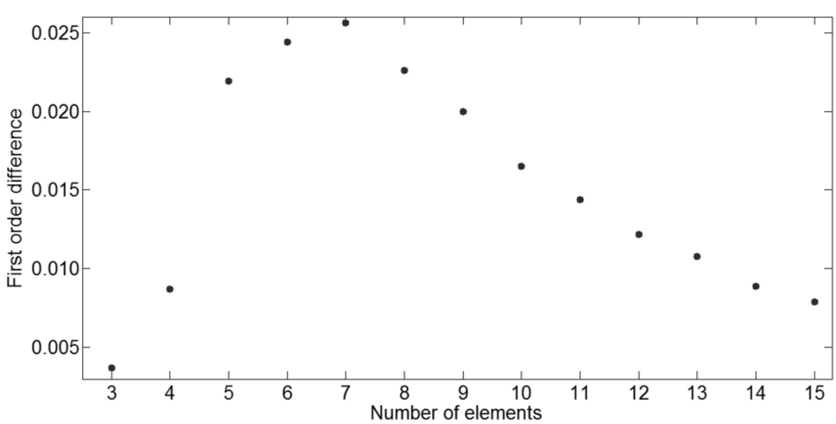

FIGURE 6. Plot of first order difference perturbations in the third column of Table 4.

With large inconsistency one cannot guarantee that none of the components of w_1 is arbitrarily small. Thus, near-consistency is a sufficient condition for stability. Note also that we need to keep the number of elements relatively small, so that the values of all the components are of the same order. The foregoing suggests that reciprocal matrices are the archetypical matrices, which produce stable eigenvectors on small perturbations of the consistent case. The conclusion is that n must be small but not too small, and one must compare *homogeneous* elements (homogeneity is one of the axioms of the AHP).

Let us now turn to the second perturbation formula. It consists of the difference between the principal eigenvectors of the perturbed matrix and the original matrix. In the current paper, we use the

"Matlab" software to simulate the second perturbation formula. Suppose the original positive reciprocal matrix is $A = [a_{ij}]_{n\times n}$ and $P = [p_{ij}]_{n\times n}$ is the perturbation matrix, both of which are produced by using random values from the Fundamental Scale. Then the calculations are as follows[6]:

(1) Calculate the principal eigenvector w_1 of A using the formula: $w_1 = \lim_{k\to\infty} \sum_{j=1}^{n} a_{ij}^{(k)} \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}$, where $a_{ij}^{(k)}$ is the (i, j) entry of the k^{th} power of the matrix A;

(2) Construct the perturbed matrix by $A' = [a_{ij}']_{n\times n} = A \circ P$, where the operation "$\circ$" is the elementwise (Hadamard) product of the matrices A and P;

(3) Calculate the principal eigenvector w_1' of the perturbed matrix A' using the formula: $w_1' = \lim_{k\to\infty} \sum_{j=1}^{n} a_{ij}^{(k)'} \Big/ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)'}$, where $a_{ij}^{(k)'}$ is the (i, j) entry of the k^{th} power of the matrix A';

(4) Calculate the perturbation vector of the principal eigenvector using the formula $\Delta w_1 = w_1' - w_1$;

(5) Run the program 50,000 times for each size matrix from 2 to 15 to do (4)—thus calculate the perturbation of the principal eigenvector;

(6) Compute the norm of each of these 50,000 perturbation vectors by taking the square root of the sums of the squares of its entries;

(7) Add all of these 50,000 norms of the vectors and form the average value, which leads to Table 4 and Figures 5 and 6.

Note the peak shown in Figure 6 in the value of the first order differences for seven elements.

Consistency of a Hierarchical System and of a Network System

The current section moves from calculating the consistency of a single pairwise comparison matrix to calculating the consistency for a hierarchy and for a network.

The Consistency of a Hierarchy

The consistency of a hierarchically constructed system of many parts and subparts is obtained[11] by first taking sums of products of each consistency index with the composite priority of its criterion. The ratio is then formed from this number with the sums of the products of the random consistency index for that order matrix with the composite priority of its criterion. In general, the ratio should be in the neighborhood of 0.10 to minimize concern for needed improvements in the judgments.

Let n_j, $j = 1, 2, \ldots, h$ be the number of elements in the j^{th} level of the hierarchy. Let w_{ij} be the composite weight of the i^{th} criterion of the j^{th} level, and let $\mu_{i,j+1}$ be the consistency index of all elements in the $(j+1)^{st}$ level compared with respect to the i^{th} criterion of the j^{th} level. The consistency index of a hierarchy is given by $C_H = \sum_{j=1} \sum_{i=1} w_{ij} \mu_{i,j+1}$, where $w_{ij} = 1$ for $j = 1$, and $n_{i,j+1}$ is the number of elements of the $(j+1)^{st}$ level with respect to the i^{th} criterion of the j^{th} level.

The Inconsistency of a System

We want to represent both the inconsistency along paths beginning with a goal and the inconsistency in cycles. For paths, we want the initial priorities of the elements. For cycles, we want the limit priorities of the elements. We need to evaluate inconsistency by the weight of the corresponding elements. Also, we need the influence priority of an element of a component to compare elements in another component. In the end, we need to weight by the priorities K_C of the supercriteria in the control hierarchy:

$$C_s = \sum_{\substack{control\\criteria}} K_c \sum_{\substack{all\\chains}} \left(\sum_{j=1}^{h} \sum_{i=1}^{n_{ij+1}} w_{ij}\mu_{ij+1} + \sum_{\substack{control\\criteria}} K_c \sum_{k=1}^{s} \sum_{j=1}^{n_k} w_{jk} \sum_{h=1}^{|c_h|} w_{(k)(h)\mu_k(j,h)}\right)$$

where $n_j = j = 1,2,\ldots,h$ is the number of elements in the j^{th} level and μ_{ij+1} is the consistency index of all elements in the $(j+1)^{st}$ level with respect to the i^{th} criterion of the j^{th} level. In the second term, $w_{(k)(h)}$ is the priority of the influence of the h^{th} component on the k^{th} component, and w_{jk} is the limit priority of the j^{th} element in the k^{th} component. In the case of a hierarchy, there are no cycles and the second term is equal to zero. As in the measurement of consistency of a hierarchy, this index must be divided by the corresponding index with random inconsistencies.

In both hierarchies and networks, it can be shown that the inconsistency cannot be worse than that of the inconsistency of the most inconsistent subset whose functions are also pairwise compared.[11]

More Examples

Here are a few examples to illustrate the foregoing idea that 7 or 8 is a natural bound on the number of interacting elements imposed by the need for consistency. Although examples are not a proof, still we present them here because it is interesting how often that bound comes up.

Number of Jurors

We begin with a social system example about people's interdependent judgments. Useful observations about how many jurors is the best number has been extensively studied over the years by many people. Condorcet's Jury Theorem[12] says that a larger jury, on average, reaches a more accurate decision. However, Bag, Levine, and Spencer[13] of the University of Surrey have shown that with a larger jury size, the probability of reaching a correct verdict may, in fact, decrease, contrary to the Condorcet Jury Theorem. They showed that if the jurors coordinate on any one of a number of (equally plausible) asymmetric equilibriums other than the symmetric equilibrium,[14] the probability of accuracy reaches a maximum for a particular jury size and remains unchanged with larger juries. In referring to a part of their research and statistical findings, Nagel and Neef[15] write:

> . . . the most important aspect is the point at which the weighted sum of errors is least. This point is reached at a jury size somewhere between six and eight; the nearest whole number is seven. The model therefore predicts that a jury of seven members will minimize errors in the fashion we assume would be optimum. Subject to the limitations on the coin-flipping model . . . , we can refer to a seven-member jury as the optimum jury size for unanimous juries.

Stimulus-response Theory

This example is from psychology. G. A. Miller [9] wrote about the magic number 7 ± 2. He observed that in responding to successive stimuli, performance is nearly perfect up to 7 different stimuli but declines as the number of different stimuli is increased. He says that the memory span of young adults is approximately seven items. He also concluded that memory span is not limited in terms of bits but rather in terms of chunks. A chunk is the largest meaningful unit in the presented material that the person recognizes.

Maslow's Human Needs

This example is also from psychology. The following 7 human needs were identified by Maslow[17]: the "Basic needs or Physiological needs" of a human being, "Safety and security needs," "Love and Belonging" needs, "Esteem" level needs, "Cognitive" level needs, "Aesthetic" level needs, and finally, at the top of the pyramid, the "Need for Self-actualization."

Music

The numbers 7 or 8 again play an important role in music in which octaves (Latin: *octavus*, or "eighth") notes are grouped together. The octave relationship is a natural phenomenon that has been referred to as the "basic miracle of music," the use of which is common in most musical systems.

Computer Science

In computer science, we do not have more than 7 to 8 interacting component functions in spite of the complexity of today's computers and network systems. For example in the Von Neumann architecture, according to functions we have: (1) input devices; (2) output devices; (3) the control unit; (4) the arithmetic unit, composing the central processing unit; and (5) the memory unit. Even when going into greater detail, the hardware of a personal computer is composed of a case, a power supply unit, the motherboard, expansion cards, peripheral devices, storage devices, and input/output devices. Similarly, microcomputer design includes the microprocessor, read and write memory (RAM), read only memory (ROM), Input/Output unit, address bus, data bus, and control bus (totaling 7 elements) in modern micro-computer architecture.[18] This does not mean that there are not millions of parts, but rather that they are always hierarchically structured and grouped in no more than 7 or 8 interacting components. We have millions of bytes of data or segments in a hard disk; however, they are not interacting parts but rather parts of a whole, as are the billions of neurons in the brain. The same thing applies to networks where we can have thousands of interconnected devices, but they are not actually communicating altogether—they are used as intermediates in a hierarchical structure, even in the case of a fully connected network, as are mesh networks. The byte itself was designed as a unit of information, consisting of 8 binary digits (bits).

Biology

The next two examples are from biology. Animal cell functions and organelles are linked to each other for the overall behavior of the cell. The cell has the following organelles: Golgi Apparatus, Lysosomes, Mitochondria, Ribosomes, Endoplasmic Reticulum, Vacuole, and protein receptors to bring in needed material to and take waste out of the cell. In Eukaryotic cells, all organelles are controlled by the centrally located Nucleus. As we said before, they communicate through the Cytoplasm. Plant cells also have Chloroplasts for making Chlorophyll.

Our hierarchic body subsystems are also a good example of this principle. We have the (1) Cardiovascular or Circulatory System; (2) Respiratory System; (3) Digestive System; (4) Endocrine System; (5) Reproductive System; (6) Nervous System; (7) Muscular System; (8) Skeletal System; and (9) Integumentary System (i.e., skin, hair, nails, sweat glands). The last three systems provide the framework for support and movement, and the brain acts as the controller.

Mechanical Engineering

The next example is from mechanical engineering. John Newman of Vintage Emperor Clock Consultant, THE VILLAGE CLOCKSMITH, Old Prattville, Prattville, Alabama, answered the question: What is the maximum number of wheels in a manufactured clock or watch that keeps time only without chimes or strike? He answered that usually, there are 5 or 6 wheels included.

Biblical Example

Finally, 7 is a number of great significance in the Bible, as one reviewer of this paper learned in a recent trip to Israel. God created the world in 6 days and rested the seventh day. God, being God, neither required 6 days to create anything nor to rest afterward, but He may have wanted to give us a pattern; work days and a rest day are independent of each other. This 7-day pattern is so important that He explicitly commands in the Fourth Commandment: "Six Days you shall work, but on the seventh day you shall rest" (Exodus 34:21). In general, the number 7 in the Bible represents "divine perfection, totality or completion and is mentioned at least 490 times."[19]

Conclusions

In this paper, systems of elements are considered that are interdependent, mutually interact, and compensate for the movement of other elements. The forces that cause such interaction can be gravitational, electromagnetic, mechanical, or mental.

It was shown above that there is a limit to the number of elements that can work together interdependently without breakdown in their cooperative effort. Every system, including the human body, consists of a hierarchy of parts, subparts, and still smaller parts. It should always be possible to identify a part when it becomes inconsistent with the workings of the other parts. As the system ages, some of its parts weaken more than the other parts, and if it is very large, the system would have difficulty identifying the defective parts. It has been demonstrated that 7 or 8 is a limit on the number of interdependent elements working together in a module of a system.

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